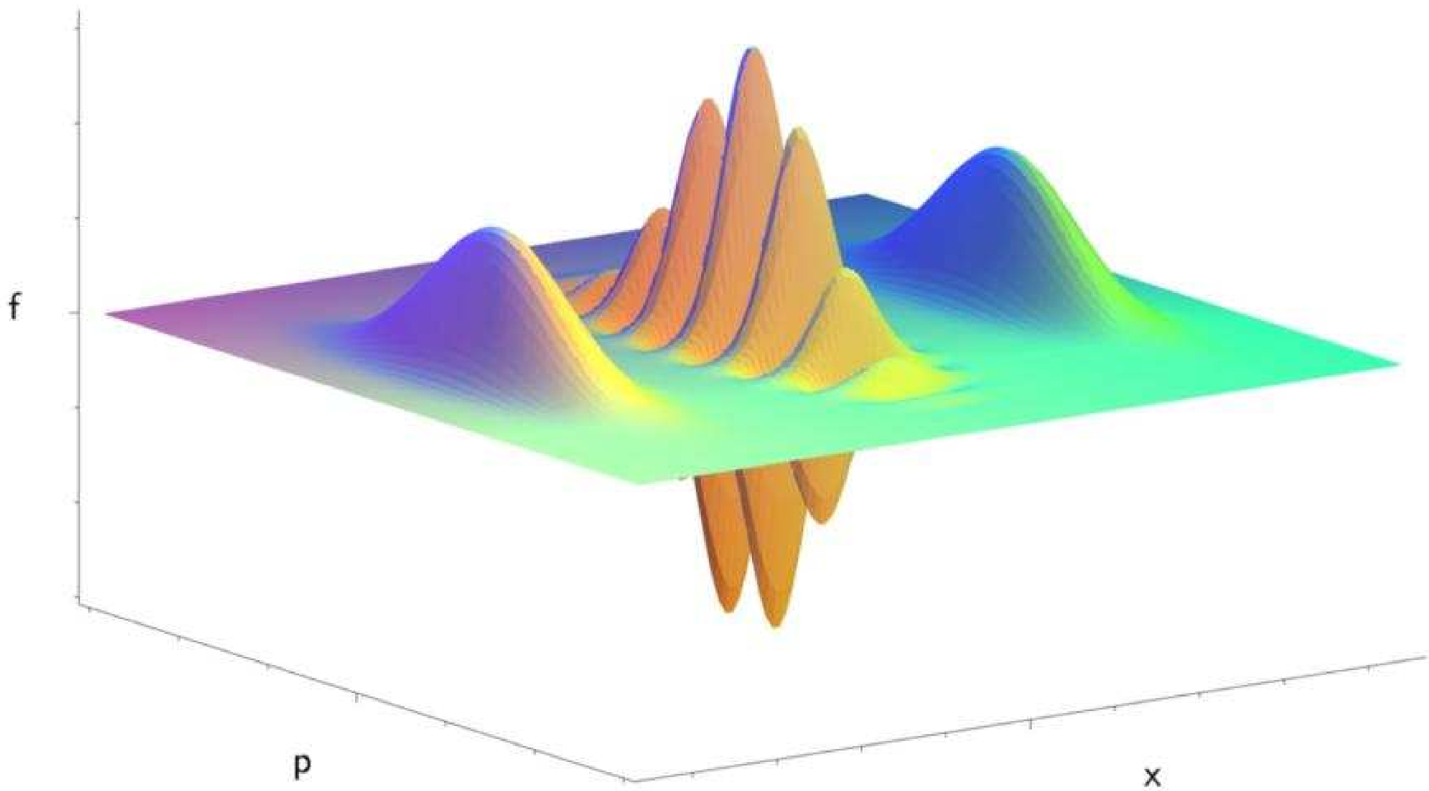
DOUBLE SLIT



QUANTUM MECHANICS LIVES AND WORKS IN PHASE-SPACE

The Wigner phase-space quasi-probability distribution function

Three alternate paths to quantization:
1. Hilbert space (Heisenberg, Schrödinger, Dirac)
2. Path integrals (Dirac, Feynman)
3. Wigner's phase-space distribution function (Wigner 1932; Groenewold 1946; Moyal 1949; Baker 1958; Fairlie 1964; ...)

$$f(x,p) = \frac{1}{2\pi}\int dy\ \psi^*(x - \frac{\hbar}{2}y)\ e^{-iyp}\psi(x + \frac{\hbar}{2}y).$$

A special representation of the density matrix (Weyl correspondence).

Useful in describing **quantum** transport/flows in phase space $\rightsquigarrow$ quantum optics; nuclear physics; study of decoherence (eg, quantum computing)

But also signal processing (time-frequency analysis); Intriguing mathematical structure of relevance to Lie Algebras, M-theory,...

A complete, autonomous formulation of QM based on c-number functions in phase-space, which compose through a special operation.

$$f(x,p) = \frac{1}{2\pi} \int dy\, \psi^*(x - \frac{\hbar}{2} y)\, e^{-iyp} \psi(x + \frac{\hbar}{2} y).$$

Normalized, $$\int dp dx\, f(x,p) = 1\,.$$

- Real

- Bounded: $-\frac{2}{h} \leq f(x,p) \leq \frac{2}{h}$ (Schwarz Inequality)

- p- or x-projection leads to marginal probability densities: A spacelike shadow $\int dp\, f(x,p) = \rho(x)$; **or else** a momentum-space shadow $\int dx\, f(x,p) = \sigma(p)$, resp; both positive semidefinite. But cannot be conditioned on each other. The uncertainty principle is fighting back $\rightsquigarrow$

- f can, and most often does go negative (Wigner). A hallmark of **interference**. "Negative probability" (Bartlett; Moyal; Feynman; Bracken & Melloy).

Smoothing f by a filter of size larger than $\hbar$ (eg, convolving with phase-space Gaussian) results in a positive semidefinite function: it has been coarsened to a classical distribution (Cartwright, 1975).

When is a real $f(x,p)$ a bona-fide Wigner function?
When its Fourier transform L-R-factorizes

$$\tilde{f}(x,y) = \int dp\; e^{ipy} f(x,p) = g_L^*(x - \hbar y/2)\; g_R(x + \hbar y/2)\;,$$

$\left(\frac{\partial^2 \ln \tilde{f}}{\partial(x-\hbar y/2)\partial(x+\hbar y/2)} = 0\right)$, so $g_L = g_R$ from reality.

• Nevertheless, it **is** a distribution: it yields expectation values from phase-space c-number functions.

Given an operator $\mathbf{A}(\mathbf{x},\mathbf{p})$ in Weyl's association rule (1927), $= \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\; A(x,p)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x))$, the corresponding phase-space function $A(x,p)$ obtained by $\mathbf{p}\mapsto p$, $\mathbf{x}\mapsto x$ yields that operator's expectation value

$$\langle \mathbf{A} \rangle = \int dx dp\; f(x,p)\; A(x,p).$$

Dynamical evolution of f (Moyal):

Liouville's Thm, $\partial_t f + \{f, H\} = 0$, quantum generalizes to

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar}\;,$$

based on the $\star$-product (Groenewold):

$$\boxed{\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\,,}$$

the essentially unique one-parameter ($\hbar$) associative deformation of Poisson Brackets of classical mechanics. (viz. $\hbar \to 0$).
(Isomorphism: $\mathbf{AB} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp (A\star B)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x))$.)

Systematic solution of time-dependent equations is usually predicated on the spectrum of stationary ones. But time-independent pure-state Wigner functions $\star$-commute with H.

However, they further obey a more powerful functional $\star$-genvalue equation (Fairlie, 1964):

$$H(x,p) \star f(x,p) = H(x + \frac{i\hbar}{2}\vec{\partial}_p, p - \frac{i\hbar}{2}\vec{\partial}_x)\, f(x,p)$$
$$= f(x,p) \star H(x,p) = E\; f(x,p)\;,$$

which amounts to a complete characterization of them:

For real functions $f(x,p)$, the Wigner form is equivalent to compliance with the $\star$-genvalue equation ($\Re$ and $\Im$ parts).
(Curtright, Fairlie, & Zachos, Phys Rev **D58** (1998) 025002)

$\Longrightarrow$ Projective orthogonality spectral properties

$$f \star H \star g = E_f\; f \star g = E_g\; f \star g.$$

For $E_g \neq E_f$, $\Longrightarrow$ $f \star g = 0.$

Precluding degeneracy, for $f = g$,

$$f \star H \star f = E_f\; f \star f = H \star f \star f,$$

$$\Longrightarrow \qquad f \star f \propto f.$$

fs $\star$-project onto their space.

$$\boxed{f_a \star f_b = \tfrac{1}{h}\, \delta_{a,b}\, f_a.}$$

• The normalization matters (Takabayasi, 1954): despite linearity of the equations, it prevents superposition of solutions (this is not how QM interference works here!).

$$\int dpdx\ f \star g = \int dpdx\ f g,$$

so, for different $\star$-genfunctions,

$$\int dpdx\ f g = 0.$$

$\rightsquigarrow$ Negative values are a feature, not a liability.

NB $\rightsquigarrow$ $$\int H(x,p) f(x,p)\ dxdp = E \int f\ dxdp\ .$$

NB $\rightsquigarrow$ $$\int f^2\ \ dxdp = \tfrac{1}{h}\ .$$

• For any function, $\langle |g|^2 \rangle$ need not ≥ 0.

But $\langle g^* \star g \rangle \geq 0$ ($\rightsquigarrow$ the **uncertainty principle**).

▼ **Pf**

$$H(x,p) \star f(x,p)$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left((p - i\frac{\hbar}{2}\,\overrightarrow{\partial}_x)^2/2m + V(x)\right)\int dy\, e^{-iy(p+i\frac{\hbar}{2}\overleftarrow{\partial}_x)}\psi^*(x-\frac{\hbar}{2}y)\,\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\,\left((p - i\frac{\hbar}{2}\,\overrightarrow{\partial}_x)^2/2m + V(x+\frac{\hbar}{2}y)\right)e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\,\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\, e^{-iyp}\left((i\,\overrightarrow{\partial}_y + i\frac{\hbar}{2}\,\overrightarrow{\partial}_x)^2/2m + V(x+\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\,\psi(x+\frac{\hbar}{2}y)\\
&= \frac{1}{2\pi}\int dy\, e^{-iyp}\psi^*(x-\frac{\hbar}{2}y)\;E\;\psi(x+\frac{\hbar}{2}y) =
\end{aligned}$$

$$= E\; f(x,p);$$

• Action of the effective differential operators on ψ^* turns out to be null;

$$f \star H$$

$$= \frac{1}{2\pi}\int dy\, e^{-iyp}\left(-(\overrightarrow{\partial}_y - \frac{\hbar}{2}\,\overrightarrow{\partial}_x)^2/2m + V(x-\frac{\hbar}{2}y)\right)\psi^*(x-\frac{\hbar}{2}y)\,\psi(x+\frac{\hbar}{2}y)$$

$$= E\; f(x,p).$$

Conversely, the pair of $\star$-eigenvalue equations dictate, for $f(x,p) = \displaystyle\int dy\; e^{-iyp}\tilde{f}(x,y)$,

$$\int dy\; e^{-iyp}\left(-\frac{1}{2m}(\overrightarrow{\partial}_y \pm \frac{\hbar}{2}\,\overrightarrow{\partial}_x)^2 + V(x \pm \frac{\hbar}{2}y) - E\right)\tilde{f}(x,y) = 0.$$

$\rightsquigarrow$ Real solutions of $H(x,p) \star f(x,p) = E\; f(x,p)$ $(= f(x,p) \star H(x,p))$ must be of the Wigner form,
$f = \displaystyle\int dy\; e^{-iyp}\psi^*(x-\tfrac{\hbar}{2}y)\psi(x+\tfrac{\hbar}{2}y)/2\pi$, (s.t. $\mathbf{H}\psi = E\psi$).

The wonderful truth (still!) sinking in: $\star$-multiplication of c-number phase-space functions is in complete isomorphism (Groenewold) to Hilbert-space operator algebra.

SIMPLE HARMONIC OSCILLATOR

Solve **directly** for $H = (p^2 + x^2)/2$
(with $\hbar = 1$, $m = 1$, $\omega = 1$):

$$\left((x + \frac{i}{2}\partial_p)^2 + (p - \frac{i}{2}\partial_x)^2 - 2E \right) f(x,p) = 0.$$

Mere PDEs! Imaginary part: $(x\partial_p - p\partial_x) f = 0$. $\rightsquigarrow$
f depends on only one variable, $z = 4H = 2(x^2 + p^2)$. $\rightsquigarrow$

$$\left(\frac{z}{4} - z\partial_z^2 - \partial_z - E \right) f(z) = 0.$$

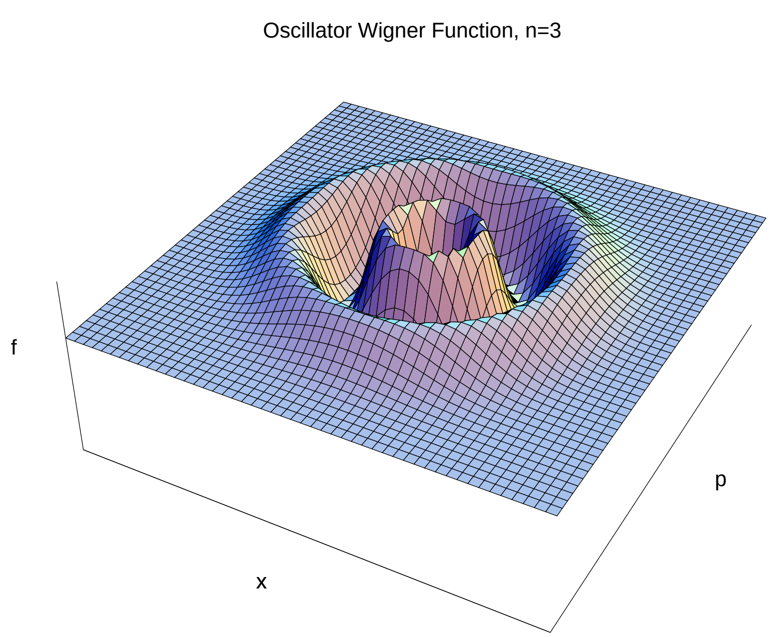
Set $f(z) = \exp(-z/2) L(z)$ $\Longrightarrow$ Laguerre's eqn

$$\left(z\partial_z^2 + (1 - z)\partial_z + E - \frac{1}{2} \right) L(z) = 0.$$

Satisfied by Laguerre polynomials, $L_n = e^z \partial^n (e^{-z} z^n)/n!$, for $n = E - 1/2 = 0, 1, 2, \ldots$ $\rightsquigarrow$ eigen-Wigner-functions are

$$f_n = \frac{(-1)^n}{\pi} e^{-2H} L_n(4H); \qquad L_0 = 1, \quad L_1 = 1 - 4H,$$

$L_2 = 8H^2 - 8H + 1, \ldots$ $\qquad \lozenge$ not positive definite.



$(\Sigma_n f_n = 1/2\pi.)$

Dirac's Hamiltonian factorization for algebraic solution **carries through intact in $\star$ space**:

$$H = \frac{1}{2}(x - ip) \star (x + ip) + \frac{1}{2} \,,$$

so define

$$a \equiv \frac{1}{\sqrt{2}}(x + ip), \qquad\qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(x - ip).$$

$$a \star a^\dagger - a^\dagger \star a = 1 \,.$$

$\star$-Fock vacuum:

$$a \star f_0 = \frac{1}{\sqrt{2}}(x + ip) \star e^{-(x^2+p^2)} = 0 \,.$$

Associativity of the $\star$-product permits the customary ladder spectrum generation; $H \star f = f \star H$ $\star$-genstates:

$$f_n \propto (a^\dagger \star)^n \, f_0 \, (\star a)^n \,.$$

• real, like the Gaussian ground state;

$\rightsquigarrow$ left-right symmetric;

$\star$-orthogonal for different eigenvalues;

project to themselves, since the Gaussian ground state does, $f_0 \star f_0 \propto f_0$.

TIME EVOLUTION

Isomorphism to operator algebras $\rightsquigarrow$ associative combinatoric operations completely analogous to Hilbert space QM

$\rightsquigarrow$ $\star$-unitary evolution operator, a "$\star$-exponential" (BFFLS)

$$U_\star(x,p;t) = e_\star^{itH/\hbar} \equiv$$

$$1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!} H \star H + \frac{(it/\hbar)^3}{3!} H \star H \star H + ...,$$

$$f(x,p;t) = U_\star^{-1}(x,p;t) \star f(x,p;0) \star U_\star(x,p;t).$$

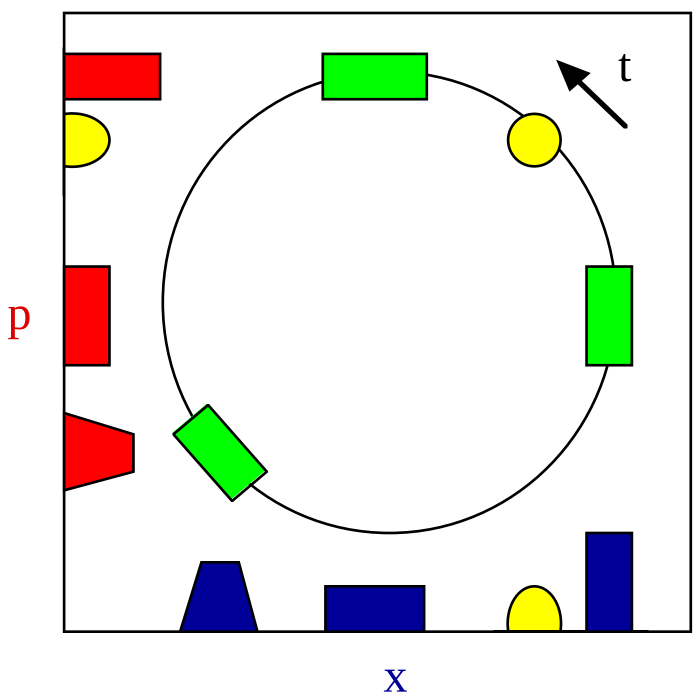
NB Collapse to **classical** trajectories,

$$\frac{dx}{dt} = \frac{x \star H - H \star x}{i\hbar} = \partial_p H = p \; ,$$
$$\frac{dp}{dt} = \frac{p \star H - H \star p}{i\hbar} = -\partial_x H = -x \quad \Longrightarrow$$
$$x(t) = x \cos t + p \sin t,$$
$$p(t) = p \cos t - x \sin t.$$

$\Longrightarrow$ For SHO the functional form of the Wigner function is preserved along classical phase-space trajectories (Groenewold, 1946):

$$f(x,p;t) = f(x\cos t - p\sin t, p\cos t + x\sin t; 0).$$

Any Wigner distribution rotates uniformly on the phase plane around the origin, essentially classically, even though it provides a complete quantum mechanical description.



NB In general, loss of simplicity upon integration in x (or p) to yield probability densities: the rotation induces shape variations of the oscillating probability density profile.

Only if (eg, coherent states) a Wigner function configuration has an additional axial $x-p$ symmetry around its *own* center, will it possess an invariant profile upon this rotation, and hence a shape-invariant oscillating probability density.

THE WEYL CORRESPONDENCE BRIDGE

Weyl's correspondence map, by itself, merely provides **a change of representation between phase space and Hilbert space**
$\leadsto$ Helps contrast classical to quantum mechanics on common footing.

$$\mathbf{A}(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\; a(x,p)\exp(i\tau(\mathbf{p}-p)+i\sigma(\mathbf{x}-x)),$$

Inverse map (Wigner):

$$a(x,p) = \frac{1}{2\pi}\int dy\; e^{-iyp}\left\langle x+\frac{\hbar}{2}y\right|\mathbf{A}(\mathbf{x},\mathbf{p})\left|x-\frac{\hbar}{2}y\right\rangle \;.$$

$$\begin{array}{ccc}
\text{PHASE SPACE} & & \text{HILBERT SPACE} \\
a & \xrightarrow{Weyl} & \mathbf{A} \\
{\scriptstyle quantum}\downarrow & & \downarrow{\scriptstyle quantum} \\
a\star b & \xrightarrow{Groenewold} & \mathbf{AB} \\
{\scriptstyle classical\ \hbar=0}\downarrow & & \downarrow{\scriptstyle Bracken\ \hbar=0} \\
ab & \xrightarrow{Weyl} & \mathbf{A}\odot\mathbf{B}
\end{array}$$

$\leadsto$ A plethora of choice-of ordering quantum mechanics problems reduce to purely $\star$-product algebraic ones: all deformations (ordering choices) can be surveyed systematically in phase space.
(Curtright & Zachos, New J Phys **4** (2002) 83.1-83.16
[hep-th/0205063])

A historical footnote on the landmark publications on phase-space quantization
Implicitly, the bulk of the formulation is contained in Groenewold's and Moyal's seminal papers. But this has been a slow story of emerging connections and chains of ever sharper reformulations, and confidence in the autonomy of the formulation arose very slowly. As a result, attribution of critical milestones cannot avoid subjectivity: it cannot automatically highlight merely the *earliest* occurence of a construct, unless that has also been compelling enough to yield an "indefinite stay against confusion" about the logical structure of the formulation.

H Weyl, Z Phys **46** (1927) 1
(also reviewed in his book, *The Theory of Groups and Quantum Mechanics*, Dover, New York, 1931) introduces the correspondence of "Weyl-ordered" operators to phase-space (c-number) kernel functions (as well as discrete QM application of Sylvester's (1883) clock and shift matrices).

J von Neumann, Math Ann **104** (1931) 570-578,
in a technical aside off a study of the uniqueness of Schrödinger's representation, includes an implicit version of the $\star$-product which promotes Weyl's correspondence rule to full isomorphism between Weyl-ordered operator multiplication and $\star$-convolution of kernel functions.

E Wigner, Phys Rev **40** (1932) 749
introduces the eponymous phase-space distribution function controlling quantum mechanical diffusive flow in phase space, on the basis of intuitive arguments. It specifies the time evolution of this function and applies it to quantum statistical mechanics.

H Groenewold, Physica **12** (1946) 405-460
A seminal but somewhat unappreciated paper which fully understands the Weyl correspondence and produces the WF as the classical kernel of the density matrix. It reinvents and streamlines von Neumann's construct into the standard $\star$-product, in a systematic exploration of the isomorphism between Weyl-ordered operator products and their kernel function compositions. It further works out the harmonic oscillator WF.

J Moyal, Proc Camb Phil Soc **45** (1949) 99-124 amounts to a grand synthesis:
It establishes an independent formulation of quantum mechanics in phase space. It systematically studies all expectation values of Weyl-ordered operators, and identifies the Fourier transform of their moment-generating function (their characteristic function) to the Wigner Function. It further interprets the subtlety of the "negative probability" formalism and reconciles it with the uncertainty principle. Not least, it recasts the time evolution of the Wigner Function through a deformation of the Poisson Bracket into the Moyal Bracket (the commutator of $\star$-products, i.e., the Weyl correspondent of the Heisenberg commutator), and thus opens up the way for a systematic study of the semiclassical limit. Before publication, Dirac has already been impressed by this work, contrasting it to his own ideas on functional integration, in Bohr's Festschrift (P A M Dirac, Rev Mod Phys **17** (1945) 195-199).

T Takabayasi, Prog Theo Phys **11** (1954) 341-373
investigates the fundamental projective normalization condition for pure state Wigner functions, and

exploits Groenewold's link to the conventional density matrix formulation. It further illuminates the diffusion of wavepackets.

G Baker, Phys Rev **109** (1958) 2198-2206
envisions the logical autonomy of the formulation, based on postulating the projective normalization condition. It resolves measurement subtleties in the correspondence principle and appreciates the significance of the anticommutator of the $\star$-product as well, thus shifting emphasis to the $\star$-product itself, over and above its commutator.

D Fairlie, Proc Camb Phil Soc **60** (1964) 581-586 (also see W Kundt, Z Nat Forsch **a22** (1967) 1333-6; L Cohen, Jou Math Phys **17** (1976) 1863; J Dahl, pp 557-571 in *Energy Storage and Redistribution* (J Hinze (ed), Plenum Press, New York, 1983))
explores the time-independent counterpart to Moyal's evolution equation, which involves the $\star$-product, beyond mere Moyal Bracket equations, and derives (instead of postulating) the projective orthonormality conditions for the resulting Wigner functions. These now allow for a unique and full solution of the quantum system, in principle (without any reference to the conventional Hilbert-space formulation). Thus, autonomy of the formulation is fully recognized.

M Berry, Philos Trans R Soc London A287 (1977) 237
elucidates the subtleties of the semiclassical limit, ergodicity, integrability, and the singularity structure of Wigner function evolution.

F Bayen, M Flato, C Fronsdal, A Lichnerowicz, and D Sternheimer, Ann Phys **111** (1978) 61-110; *ibid* 111-151, analyzes systematically the deformation structure and the uniqueness of the formulation, and consolidates it mathematically. It provides explicit solutions to standard problems and introduces influential technical innovations, such as the $\star$-exponential.

T Curtright, D Fairlie, and C Zachos, Phys Rev **D58** (1998) 025002
demonstrates more directly the equivalence of the time-independent $\star$-genvalue problem to the Hilbert space formulation, and hence its logical autonomy; formulates Darboux isospectral systems in phase space; establishes the covariant transformation rule for general *nonlinear* canonical transformations (with reliance on the classic work of P Dirac, Phys Z Sowjetunion **3** (1933) 64); and thus furnishes explicit solutions of nontrivial practical problems on first principles, without recourse to the Hilbert space formulation. Efficient techniques, e.g. for perturbation theory, are based on generating functions for complete sets of Wigner functions in T Curtright, T Uematsu, and C Zachos, J Math Phys **42** (2001) 2396-2415.

M Hug, C Menke, and W Schleich, Phys Rev **A57** (1998) 3188-3205; *ibid* 3206-3224
introduce techniques for numerical solution of $\star$-equations on a basis of Chebyshev polynomials.

A writeup close to this talk found in C Zachos, Int J Mod Phys **A17** (2002) 297-316 [hep-th/0110114] and a forthcoming book (World Scientific, 2004) by Zachos, Fairlie and Curtright, "Quantum Mechanics in Phase Space"